Topological Phase Signalling Theorem

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Abstract

This manuscript presents a precise mathematical statement and constructive proof of a phenomenon in quantum foundations: when global transformations depend functionally on the global state, local pre-operations on a spatially separated subsystem can produce observable changes in another subsystem's reduced state. This result isolates the minimal structural ingredients for state-dependent topological signalling.

We demonstrate that a state-dependent global unitary, with a phase functional $\phi[\rho]$ sensitive to subsystem A and a generator $\hat{G}$ coupling subsystem B to an auxiliary system F, yields a different reduced state on B based solely on prior local actions on A. A constructive three-qubit example provides an explicit closed-form expression for the trace distance between outcomes.

The work distinguishes between mathematical formalism and physical realisability, identifying which assumptions require additional constraints—such as causality axioms or thermodynamic consistency—for laboratory implementation. We offer a toolkit comprising: (i) a theorem on partial-trace invariance failure; (ii) a toy model quantifying the effect; (iii) conditions under which the phenomenon vanishes. This theorem identifies how operational signalling manifests, challenging researchers to define the specific physical axioms that forbid such state-dependent structures or to explore their experimental quantification through engineered feedback loops.

1 keywords

Quantum mechanics, Topological signalling, State-dependent unitaries, Partial-trace invariance, Quantum foundations, Local operations, Trace distance, Tripartite systems, Phase functional, No-signalling.

2 Setup and definitions

Definition 2.1 (Tripartite system). *Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_F$ be a finite-dimensional Hilbert space describing three subsystems labeled A, B and F. Denote by $\mathcal{D}(\mathcal{H})$ the set of density operators on $\mathcal{H}$.[1]*

Definition 2.2 (State-dependent global unitary). *A map $\mathcal{U} : \mathcal{D}(\mathcal{H}) \to \mathcal{U}(\mathcal{H})$ is a* state-dependent global unitary *if for each $\rho \in \mathcal{D}(\mathcal{H})$ there is a unitary operator*

$$U(\rho) = \exp\big(-i\,\phi[\rho]\,\hat{G}\big) \tag{1}$$

where $\hat{G} = \hat{G}_{BF}$ acts nontrivially only on $\mathcal{H}_B \otimes \mathcal{H}_F$ (and as identity on $\mathcal{H}_A$), and $\phi[\cdot] : \mathcal{D}(\mathcal{H}) \to \mathbb{R}$ is a real-valued functional of the global state as shown in eq.(1).

Definition 2.3 (Protocol). *Fix initial $\rho_0 \in \mathcal{D}(\mathcal{H})$ and let V_A be a local operation acting only on A (unitary or CPTP with Kraus operators supported on $\mathcal{H}_A$). The protocol is (eq.2):*

$$\rho' = (V_A \otimes \mathbb{I}_{BF})\,\rho_0\,(V_A^\dagger \otimes \mathbb{I}_{BF}), \qquad \rho_{\text{out}} = U(\rho')\,\rho'\,U(\rho')^\dagger. \tag{2}$$

The reduced state on B after the protocol is $\rho_B = \mathrm{Tr}_{AF}[\rho_{\text{out}}]$.[3]

3 Main theorem

Theorem 3.1 (Topological Phase Signalling Theorem). *Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_F$ be finite-dimensional. Suppose:*

1. *The global unitary is state-dependent in the sense $U(\rho) = \exp\Big(-i\phi[\rho]\hat{G}\Big)$ with $\hat{G}$ acting only on BF and $\phi[\rho]$ a non-constant functional that depends (nontrivially) on the reduced A-statistics of ρ.*

2. *There exist two distinct local operations $V_A \neq V'_A$ acting only on subsystem A (unitary or CPTP with Kraus operators supported on $\mathcal{H}_A$) such that (eq.3)*

$$\phi\big[(V_A \otimes \mathbb{I}_{BF})\rho_0(V_A^\dagger \otimes \mathbb{I}_{BF})\big] \quad \neq \quad \phi\big[(V'_A \otimes \mathbb{I}_{BF})\rho_0(V'^\dagger_A \otimes \mathbb{I}_{BF})\big]. \tag{3}$$

3. *The generator $\hat{G}_{BF}$ does not act trivially on B (i.e. it couples B and F in a way that $[\hat{G}_{BF},\, \mathbb{I}_B \otimes X_F] \neq 0$ for some X_F).*

Then there exist choices of ρ_0 and local operations V_A, V'_A such that the final reduced states on B differ:

$$\rho_B^{(V)} \;=\; \mathrm{Tr}_{AF}\big[\,U(\rho')\rho' U(\rho')^\dagger\big] \quad\neq\quad \rho_B^{(V')} \;=\; \mathrm{Tr}_{AF}\big[\,U(\rho'')\rho'' U(\rho'')^\dagger\big], \tag{4}$$

where $\rho' = (V_A \otimes \mathbb{I}_{BF})\rho_0(V_A^\dagger \otimes \mathbb{I}_{BF})$ and ρ'' is defined analogously for V'_A as shown in eq.(4). Equivalently, the partial trace over AF is not invariant under the choice of local operation applied on subsystem A.

Constructive proof. We give an explicit finite-dimensional construction (three qubits) demonstrating the claim.

Let $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_F \cong \mathbb{C}^2$. Denote Pauli matrices on each subsystem by $\hat{X}, \hat{Y}, \hat{Z}$ with obvious subscripts. Define the phase functional (eq.5)

$$\phi[\rho] \;=\; g\,\mathrm{Tr}\big[(\hat{X}_A \otimes \mathbb{I}_{BF})\,\rho\big], \qquad g \in \mathbb{R}\setminus\{0\}. \tag{5}$$

Choose the generator (eq.6)

$$\hat{G} \;=\; \hat{Z}_B \otimes \hat{X}_F \tag{6}$$

which acts nontrivially on BF and satisfies $\hat{G}^2 = \mathbb{I}_{BF}$.

Pick initial product state

$$\rho_0 \;=\; |0\rangle\langle 0|_A \otimes |+\rangle\langle +|_B \otimes |0\rangle\langle 0|_F\,, \tag{7}$$

where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ (eq.7).

Consider two local actions on A (eq.8):

$$V_A = \mathbb{I}_A, \qquad V'_A = H_A \tag{8}$$

where H is the Hadamard gate; $H|0\rangle = |+\rangle$.

Compute the phase values (eq.9):

$$\phi[\rho'] = g\,\mathrm{Tr}(\hat{X}_A|0\rangle\langle 0|) = 0, \qquad \phi[\rho''] = g\,\mathrm{Tr}(\hat{X}_A|+\rangle\langle +|) = g. \tag{9}$$

Thus the two global unitaries are (eq.10)

$$U(0) = e^{-i0\hat{G}} = \mathbb{I}, \qquad U(g) = e^{-ig\hat{G}} = \cos(g)\,\mathbb{I} - i\sin(g)\,\hat{G}, \tag{10}$$

using $\hat{G}^2 = \mathbb{I}$.

Evolve the two post-local-action states (eq.11):

$$\rho' = \rho_0 \quad (\text{since } V_A = \mathbb{I}), \qquad \rho'' = (H_A \otimes \mathbb{I}_{BF})\,\rho_0\,(H_A^\dagger \otimes \mathbb{I}_{BF}). \tag{11}$$

Case 1 (eq.12): $V_A = \mathbb{I}$. Since $U(0) = \mathbb{I}$, the final state equals ρ', and the reduced state on B is

$$\rho_B^{(V)} = \mathrm{Tr}_{AF}[\rho'] = |+\rangle\langle+|_B\,. \tag{12}$$

Case 2 (eq.13): $V_A' = H$. We must compute

$$\rho''_{\text{out}} = U(g)\,\rho''\,U(g)^\dagger. \tag{13}$$

Because ρ'' factorizes as $|+\rangle\langle+|_A \otimes |+\rangle\langle+|_B \otimes |0\rangle\langle 0|_F$, and $U(g)$ acts as identity on A, we may focus on the action on BF of $U(g)$ applied to $\sigma_{BF} := |+\rangle\langle+|_B \otimes |0\rangle\langle 0|_F$.

Compute the (pure) vector as shown in eq, (14)

$$|\psi_{BF}\rangle = |+\rangle_B \otimes |0\rangle_F \tag{14}$$

and its evolved version in eq.(15)

$$|\psi'_{BF}\rangle = U(g)\,|\psi_{BF}\rangle = (\cos g\,\mathbb{I} - i\sin g\,\hat{Z}_B \otimes \hat{X}_F)\,|+\rangle \otimes |0\rangle\,. \tag{15}$$

Evaluate the two terms (eq.16):

$$\mathbb{I}\,|+\rangle|0\rangle = |+\rangle|0\rangle\,, \qquad \hat{Z}_B \otimes \hat{X}_F\,|+\rangle|0\rangle = \hat{Z}\,|+\rangle \otimes |1\rangle = |-\rangle \otimes |1\rangle\,, \tag{16}$$

since $\hat{Z}\,|+\rangle = |-\rangle$. Therefore (eq.17)

$$|\psi'_{BF}\rangle = \cos g\;|+\rangle|0\rangle - i\sin g\;|-\rangle|1\rangle\,. \tag{17}$$

The reduced state on B is (eq.18)

$$\rho_B^{(V')} = \mathrm{Tr}_F\big[\,|\psi'_{BF}\rangle\langle\psi'_{BF}|\,\big]. \tag{18}$$

Compute the partial trace explicitly. Writing $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$, after algebra one finds that $\rho_B^{(V')}$ is a qubit state with Bloch vector along x of magnitude $\cos(2g)$:

$$\rho_B^{(V')} \;=\; \tfrac{1}{2}\big(\mathbb{I}_B + \cos(2g)\,\hat{X}_B\big). \tag{19}$$

As shown in eq.(19), on the other hand $\rho_B^{(V)} = |+\rangle\langle+|_B = \frac{1}{2}(\mathbb{I} + \hat{X}_B)$.

Thus their Bloch-x components are $r_x^{(V)} = 1$ and $r_x^{(V')} = \cos(2g)$. The trace distance between the two reduced states is

$$D\big(\rho_B^{(V)}, \rho_B^{(V')}\big) = \frac{1}{2}\big|r_x^{(V)} - r_x^{(V')}\big| = \frac{1}{2}\big(1 - \cos(2g)\big) = \sin^2 g. \tag{20}$$

For any $g \notin \pi\mathbb{Z}$ this distance is strictly positive. This explicitly demonstrates $\rho_B^{(V)} \neq \rho_B^{(V')}$, proving the theorem by concrete example.(eq.20) $\square$

Remark 3.2. The construction above is minimal (three qubits) and fully constructive: it exhibits a nonzero operational difference (trace distance) of the reduced B-state controlled solely by Alice's choice of local operation on A, due to the state-dependence of the subsequent global unitary.

4 Corollaries and necessary conditions for vanishing effect

Corollary 4.1. *If the phase functional $\phi[\rho]$ is invariant under the considered local actions of A (i.e. $\phi[(V_A \otimes \mathbb{I})\rho_0(V_A^\dagger \otimes \mathbb{I})] = \phi[(V'_A \otimes \mathbb{I})\rho_0({V'_A}^\dagger \otimes \mathbb{I})]$ for all relevant V_A, V'_A), then $\rho_B^{(V)} = \rho_B^{(V')}$ for those actions.*

Corollary 4.2. *If $\hat{G}_{BF}$ acts trivially on B (i.e. $\hat{G} = \mathbb{I}_B \otimes G_F$) then $U(\rho)$ factorizes as $\mathbb{I}_B \otimes U_F(\rho)$ and the reduced B state is unchanged by $U(\rho)$; in that case the partial-trace invariance holds.*

Remark 4.3 (When the effect disappears). The phenomenon requires three ingredients simultaneously:

1. a phase/parameter that is a nontrivial functional of the global state and is sensitive to local actions on A;

2. a global unitary depending on that functional and coupling B with an auxiliary subsystem F;
3. an initial global state (or family of states) such that the induced unitaries produce distinct reduced B states.

If any ingredient is removed, the constructed counterexample (and hence general possibility) may fail.

5 Discussion and mathematical comments

The statement of Theorem 3.1 is purely mathematical. [4] It exhibits that *state-dependent* global transformations can break a standard invariance property of reduced states under local pre-transformations. Many familiar proofs of no-signalling rely on the assumption that the global evolution is independent of the global state (i.e. a fixed unitary or fixed CPTP map). The theorem shows that dropping that assumption opens a mathematical route to dependence of local reduced states on remote local choices.

This note does *not* attempt to assert that such state-dependent unitaries are physically realizable in a given physical theory without extra constraints; it merely exposes a mathematically well-defined class of constructions where the partial-trace invariance fails. Whether physical principles (energy boundedness, causality axioms, thermodynamic consistency, decoherence, or other constraints) forbid realization of the assumptions is a separate (physical) question and outside the purely mathematical result proved here.[5]

6 Concluding remarks

In this work we have formalized and proved the *Topological Phase Signalling Theorem*, providing a minimal constructive example showing that partial-trace invariance can be broken by state-dependent global unitaries. The explicit three-qubit example quantifies the effect via trace distance and illustrates the minimal ingredients required: a nontrivial state-dependent phase functional, a generator coupling B with an auxiliary subsystem, and a suitable initial global state.

While the result is purely mathematical, it opens a conceptual window: the standard no-signalling intuition in quantum mechanics depends critically on the independence of dynamics from the global state. By isolating exactly how that assumption can fail, the theorem offers a clean framework for exploring potential operational consequences, and invites rigorous discussion about which physical principles (causality, energy bounds, thermodynamic consistency) enforce or forbid such state-dependent constructions.

Future directions include exploring maximal operational differences, extending to infinite-dimensional Hilbert spaces, classifying forbidden or admissible phase functionals, and investigating engineered feedback systems where such functional dependencies could be effectively realized. The work thus provides a bridge between rigorous mathematical formalism and foundational questions in quantum theory, suggesting a controlled pathway for probing "beyond-standard" signalling scenarios in a well-defined, quantitative manner.

Declarations

Conflict of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data Availability

The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

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